

ON THE UNIQUENESS OF SOLUTIONS OF STOCHASTIC VOLTERRA EQUATIONS

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ABSTRACT. We prove strong existence and uniqueness, and Hölder regularity, of a large class of stochastic Volterra equations, with singular kernels and non-Lipschitz diffusion coefficient. Extending Yamada-Watanabe's theorem [YW71a], our proof relies on an approximation of the process by a sequence of semimartingales with regularised kernels. We apply these results to the rough Heston model, with square-root diffusion coefficient, recently proposed in Mathematical Finance to model the volatility of asset prices.

1. INTRODUCTION

This paper deals with one-dimensional Stochastic Volterra Equations (SVE) of the following type:

$$(1.1) \quad X_t = x + \int_0^t K_1(t, s) b(s, X_s) ds + \int_0^t K_2(t, s) \sigma(s, X_s) dW_s, \quad t \in [0, T],$$

where $x \in \mathbb{R}$, $T > 0$, b and σ are Borel-measurable functions and W is a Brownian motion on the canonical setup $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$. We prove strong existence and uniqueness of (1.1) for a large class of (singular) kernels, where σ is only locally $\frac{1}{2}$ -Hölder continuous.

Although Itô stochastic integration covers this type of integrands as long as they belong to $L^2(\Omega \times [0, T])$, the potential solution of (1.1) is not a semimartingale in general because the quadratic variation can be infinite, for example in the case of the fractional Brownian motion with $H < 1/2$ [Rog97]. This prevents the use of Itô calculus. On a more practical side, the solution is non-Markovian which, combined with the singularity of the kernel, restricts the use of classical numerical schemes. This property though is particularly relevant to modelling in fields where past dependence is observed.

Volterra's seminal work was concerned with population growth models with memory. Those deterministic integral equations now bearing his name subsequently blossomed in various fields including heat conduction, spread of epidemics, engineering, viscoelasticity and hydrology, see [Lin85, Chapter 2] and [Bru17, Chapter 9] and the references therein. Models of chemical reaction also gave birth to SVEs as limits of branching processes. In such models, a catalytic super-Brownian motion arises from the study of the interaction between a reactant and a catalyst [AJ18, Section 1.2.1]. Its density satisfies a stochastic partial differential equation which,

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under certain assumptions, reduces to a SVE of the type (1.1). This is explored in more details and serves as a motivation in [MT15].

Recently, empirical evidence has justified the use of SVEs and has required refined tools to make them tractable. This is in particular true in mathematical finance, where the volatility of asset prices, already known to be non-Markovian [EP01], have now been observed to feature memory properties well captured by SVEs [ALV07, BFG16, CCR12, Fuk17]. These papers also show that SVEs fit remarkably well the behaviour of implied volatility. Additionally, the study of high-frequency data carried out in [GJR18] revealed the roughness, in the sense of low Hölder regularity, of the observed time series of the instantaneous volatility. This combination showed that the log-volatility is modelled more accurately by a fractional Brownian motion (fBm) with small Hurst parameter $H \approx 0.1$ than by a classical one (where $H = 0.5$). This precisely corresponds to a driftless version of (1.1) with constant diffusion σ . Since this seminal observation, more advanced results have enlarged this new class of *rough volatility models*, in particular showing that drift and diffusion should be state dependent. One important example is the rough Heston model introduced in [EER19], and studied further in [EER18, EEFR18, AE19, AE18], where the squared volatility satisfies

$$(1.2) \quad V_t = v_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lambda(\theta - V_s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \xi \sqrt{V_s} dW_s.$$

Existence and uniqueness in the weak sense was proved in [ALP17] using the deterministic theory of resolvents associated to the convolution kernel. However, strong existence and uniqueness was so far out of reach.

Weak existence and uniqueness are sufficient in most mathematical finance applications, especially for pricing purposes. Asymptotic methods have been used extensively in order to obtain easy-to-use approximations of models, with a strong emphasis on large deviations methods. A successful approach for the latter has been set by Dupuis and Ellis [DE97], and requires strong existence and uniqueness. Our results here are therefore the first stone to build such a theory for this class of SVEs. Pathwise uniqueness is also a key ingredient to validate numerical schemes for such equations, and practical applications thereof cannot be fully justified without them [Mao94, BHY19].

The classical existence and uniqueness results for SVEs with bounded kernels are due to Berger and Mizel [BM80a, BM80b], Protter [Pro85], Pardoux and Protter [PP90]. In the latter, coefficients are considered anticipating, hence the integrals are interpreted in the Skorohod sense. This approach was also adopted by [ØZ93] while [AN97] defines them through Malliavin calculus. The first existence and uniqueness result for singular kernels was derived in [CLJ95] in the case of linear diffusion coefficient. The general SVE case was studied by Coutin and Decreusefond in [CD00], who proved strong existence and uniqueness in a concise but elegant manner for singular kernels and Lipschitz coefficients. The authors circumvented the use of the Burkholder-Davis-Gundy (BDG) inequality, the standard tool in such scheme, which is not available in this context since the stochastic integral is not a local martingale. Instead, they relied on fractional calculus and exploited the embedding of Besov spaces into spaces of Hölder continuous functions. A slight extension and a first proof for non-Lipschitz coefficients can be found in [Wan08] where the author uses the Bihari-Lasalle inequality.

It is however much more delicate to consider coefficients that are only Hölder continuous. This is even the case for diffusions, as this feature prevents the direct use of a Grönwall-type inequality. In that regard, Yamada and Watanabe’s pathwise uniqueness theorem is one of a kind, and its extension to SVEs particularly challenging because the authors relied heavily on Itô calculus. Mytnik and Salisbury [MT15]’s result seems to be the only one so far to achieve pathwise uniqueness for SVEs with Hölder continuous diffusion coefficient. Yet, the full generality of (1.1) is not attained as their drift is a deterministic bounded function and they only considered the Riemann-Liouville kernel. More importantly, σ is only allowed to be γ -Hölder continuous with $\gamma \in (\frac{1}{2\alpha}, 1)$, which cannot reach the square-root function and becomes constraining in the rough case where $\alpha \approx 0.6$. Therefore, it remains at respectable distance from the rough Heston model, for which strong uniqueness is still an open problem, as emphasised in [KRLP19]. Finally, the assumptions used in [ALP17, MT15] to prove weak existence do not fully overlap with ours, and we additionally provide strong existence and uniqueness.

Indeed we extend Yamada-Watanabe’s theorem with mild regularity assumptions on the kernels by approximating the solution to (1.1) by a sequence of semimartingales. The latter are designed with a regularised kernel $K(t + \varepsilon, s)$ which avoids the singularity on the diagonal. The convergence takes place in $L^p(\Omega)$ for some $p > 2$ and is proved using the Hölder regularity of the solution, derived by Decreasefond [Dec02] through fractional and Malliavin calculus, and represents the cornerstone of our approach. Tanaka’s formula can then be used on the semimartingales and ‘transferred’ to the solution of the SVE by passing to the limit. In particular we clarify the link between the regularity of the kernel and the Hölder continuity of the solution.

The remainder of the paper is as follows: Section 2 gathers the definition of the model and sets the notations, recalling essential tools needed here. The proofs of strong uniqueness and existence are contained in Section 3. Finally, Section 4 shows how our setup covers the rough Heston model, and presents an extension to the multidimensional case.

Notations: The letter C will denote a constant that might change from line to line. When needed, we indicate the parameters on which it depends. We shall further consider a time frame \mathbb{T} of the form \mathbb{T} for some $T > 0$. For any $p \geq 1$ we write $\mathcal{L}^p = L^p(\mathbb{T})$ and $L^p = L^p(\Omega)$.

2. STOCHASTIC VOLTERRA INTEGRALS

2.1. Regularity. We introduce the results obtained by Decreusefond [Dec02] which still represent the state-of-the-art in terms of stochastic Volterra integrals with singular kernels. For a fixed time horizon $T > 0$, we call a kernel a map $K : \mathbb{T}^2 \rightarrow \mathbb{R}$ for which both $\int_0^t K(t, s)^2 ds$ and $K(t, s)$ are finite for all $t \in \mathbb{T}$ and $s \neq t$. The associated space is defined as

$$\mathcal{A}_K := \left\{ u : \mathbb{T} \rightarrow \mathbb{R}, \{\mathcal{F}_t\} \text{ – progressively measurable,} \right. \\ \left. \text{and } \mathbb{E} \int_0^t [K(t, s)u(s)]^2 ds < \infty, \text{ for all } t \in \mathbb{T} \right\}.$$

Hence, for all $u \in \mathcal{A}_K$ the stochastic integral

$$\widetilde{M}_t^K(u) := \int_0^t K(t, s)u(s) dW_s$$

is well defined for all $t \in \mathbb{T}$ in the Itô sense. We also need the following tools:

- The Riemann-Liouville integral of $f \in \mathcal{L}^1$ is

$$I^\alpha(f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt, \quad x \geq 0, \alpha \in (0, 1),$$

and the Riemann-Liouville derivative is defined by

$$D^\alpha(f)(x) := \frac{d}{dx} I^{1-\alpha}(f)(x), \quad x \geq 0, \alpha \in (0, 1).$$

- Define $\mathcal{I}_{\alpha,p} := I^\alpha(\mathcal{L}^p)$ equipped with the norm $\|f\|_{\mathcal{I}_{\alpha,p}} := \|D^\alpha f\|_{\mathcal{L}^p}$ if $f \in I^\alpha(\mathcal{L}^p)$ and infinity otherwise. If $\alpha > \frac{1}{p}$, then $\mathcal{I}_{\alpha,p} \subset \mathcal{C}^{\alpha-\frac{1}{p}}$, the space of $(\alpha - \frac{1}{p})$ -Hölder continuous functions null at time 0.
- Let K denote the linear map associated to $K(t, s)$ by

$$(2.1) \quad Kf(t) := \int_0^t K(t, s)f(s) ds.$$

- Finally define, for $x \in (1, 2)$ and $\gamma \in (0, 1)$,

$$\theta(x) := \frac{2x}{2-x} \quad \text{and} \quad c_{\gamma,\eta} := \sup \{ \|(D^\gamma \circ K)g\|_{\mathcal{L}^{\theta(\eta)}} : \|g\|_{\mathcal{L}^\eta} = 1 \}.$$

Now let us state the first assumption:

Assumption 2.1. There exist $\eta \in (1, 2)$ and $\alpha > 1/\theta(\eta)$ for which K is continuous from \mathcal{L}^2 to $\mathcal{I}_{\alpha+1/2,2}$ and from \mathcal{L}^η to $\mathcal{I}_{\alpha,\theta(\eta)}$.

Given the space inclusions above, the assumption also implies precise Hölder regularity for the integral (2.1).

Example 2.2. The operators associated to the following kernels satisfy Assumption 2.1:

- The Riemann-Liouville kernel

$$J_H(t, s) := \frac{(t-s)_+^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}, \quad \text{with } H \in (0, 1),$$

satisfies this assumption with $\alpha = H$ and any $\eta < 2$ [Dec02, Theorem 4.1].

- The fractional Brownian motion kernel

$$K_H(t, s) = \frac{(t-s)_+^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right),$$

where F is the Gauss Hypergeometric function, also satisfies this assumption with the same parameters as in 1. [Dec02, Theorem 4.2].

Decreusefond's main result yields the Hölder regularity of the stochastic Volterra integral [Dec02, Theorem 3.1]:

Theorem 2.3 (Decreusefond). *If Assumption 2.1 holds, $r = \theta(\eta)$ and $u \in \mathcal{A}_K \cap L^r(\Omega \times \mathbb{T})$, then $\widetilde{M}^K(u)$ has a measurable version $M^K(u)$, which belongs to $\bigcap_{\gamma < \alpha} \mathcal{I}_{\gamma, r}$. Thus $M^K(u)$ is γ -Hölder continuous for all $\gamma < \alpha - \frac{1}{r}$, and, for any $\gamma < \alpha$,*

$$(2.2) \quad \left\| \|M^K(u)\|_{\mathcal{I}_{\gamma, r}} \right\|_{L^r} =: \|M^K(u)\|_{L^r(\Omega; \mathcal{I}_{\gamma, r})} \leq c_{\gamma, \eta} \|u\|_{L^r(\Omega \times \mathbb{T})}.$$

From now on, we only consider the measurable version of the stochastic integral. We also prove the following inequalities, replacing the BDG inequality:

Lemma 2.4 (BDG-type inequality). *Let $p > 2$, $K : \mathbb{T}^2 \rightarrow \mathbb{R}$ be a kernel and u progressively measurable with $M^K(u) \in L^1$. Then*

$$(2.3) \quad \mathbb{E} |M^K(u)|^p = \mathbb{E} \left[\left| \int_0^t K(t, s) u(s) dW_s \right|^p \right] \leq \mathbb{E} \left[\int_0^t |K(t, s) u(s)|^2 ds \right]^{\frac{p}{2}}.$$

If moreover $K(t, \cdot) \in \mathcal{L}^{\frac{2p}{p-2}}$ for all $t \in \mathbb{T}$, then

$$\mathbb{E} \left[\int_0^t |K(t, s) u(s)|^2 ds \right]^{\frac{p}{2}} \leq C_T \mathbb{E} \left[\int_0^t |u(s)|^p ds \right],$$

where $C_{T, p} := \sup_{t \in \mathbb{T}} \|K(t, \cdot)\|_{\mathcal{L}^{\frac{2p}{p-2}}}$ is finite.

Proof. We first recall a useful property [SS11, Chapter 1, Lemma 4.2 (ii)]. If p and q are conjugate, g integrable and $\sup \{ |\int fg|, \|f\|_{L^q} = 1, f \text{ simple} \} = M < \infty$, then $\|g\|_{L^p} = M$. Recall that by assumption $\int_0^t K(t, s) u(s) dW_s$ is integrable with respect to \mathbb{P} . Now for any $\phi \in L^2$ simple, the following calculations follow from Cauchy-Schwarz inequality, the Itô isometry, and Hölder inequality:

$$\begin{aligned} \left| \mathbb{E} \left[\int_0^t K(t, s) u(s) dW_s \phi \right] \right| &\leq \|\phi\|_{L^2} \mathbb{E} \left[\left| \int_0^t K(t, s) u(s) dW_s \right|^2 \right]^{\frac{1}{2}} \\ &= \|\phi\|_{L^2} \mathbb{E} \left[\int_0^t |K(t, s) u(s)|^2 ds \right]^{\frac{1}{2}} \\ &\leq \|\phi\|_{L^2} \mathbb{E} \left[\int_0^t |u(s)|^p ds \right]^{1/p} \mathbb{E} \left[\int_0^t |K(t, s)|^{\frac{2p}{p-2}} ds \right]^{\frac{p-2}{2p}} \\ &\leq C_{T, p} \|\phi\|_{L^2} \|u\|_{L^p(\Omega \times \mathbb{T})} < \infty. \end{aligned}$$

Since the space L^2 is dense in $L^{\frac{p}{p-1}}$, the claim follows. \square

2.2. Semimartingale approximation. We now show how to approximate a stochastic Volterra integral by a semimartingale, following [Dun11, Tha06], who perform similar approximations for Gaussian Volterra processes. For a given kernel K and $u \in \mathcal{A}_K$, we introduce, for all $\varepsilon > 0$,

$$\begin{aligned} M_t^{K, \varepsilon}(u) &:= \int_0^t K(t + \varepsilon, s) u(s) dW_s, \\ N_t^{K, \varepsilon}(u) &:= \int_0^t K(t + \varepsilon, s) u(s) ds, \quad \text{and} \quad N_t^K(u) := \int_0^t K(t, s) u(s) ds. \end{aligned}$$

Assumption 2.5. The function $t \mapsto K(t, s)$ is differentiable on $(s, T]$ for all $s \in \mathbb{T}$. K is also triangular, i.e. $K(t, s) = 0$ for all $s > t$.

All the kernels in Example 2.2 clearly satisfy this assumption. The first lemma exhibits the semimartingale property of the processes $M^{K,\varepsilon}(u)$:

Lemma 2.6. *Under Assumption 2.5, for any $\varepsilon > 0$, $u \in \mathcal{A}_K$, $M^{K,\varepsilon}(u)$ is a \mathcal{F}_t -semimartingale with decomposition*

$$M_t^{K,\varepsilon}(u) = \int_0^t K(s+\varepsilon, s)u(s) dW_s + \int_0^t \Psi_s^\varepsilon(u) ds, \quad \text{with } \Psi_s^\varepsilon(u) = \int_0^s \partial_1 [K(s+\varepsilon, r)]u(r) dW_r.$$

Proof. The lemma follows from a straightforward application of the stochastic Fubini theorem [Pro05, Theorem 65]:

$$\begin{aligned} \int_0^t \Psi_s^\varepsilon(u) ds &= \int_0^t \int_0^s \partial_1 K(s+\varepsilon, r)u(r) dW_r ds = \int_0^t \int_r^t \partial_1 K(s+\varepsilon, r) ds u(r) dW_r \\ &= \int_0^t (K(t+\varepsilon, r) - K(r+\varepsilon, r)) u(r) dW_r \\ &= M_t^{K,\varepsilon}(u) - \int_0^t K(s+\varepsilon, s)u(s) dW_s. \end{aligned}$$

□

The next proposition proves the convergence of the integrals under integrability assumptions on the process u . Furthermore, it yields an explicit rate of convergence.

Proposition 2.7. *Let Assumptions 2.1 and 2.5 hold and let $r := \theta(\eta) > 2$.*

- (i) *If $u \in \mathcal{A}_K \cap L^r(\Omega \times \mathbb{T})$, the process $M^{K,\varepsilon}(u)$ converges to $M^K(u)$ in L^r as ε tends to 0, uniformly in $t \in \mathbb{T}$.*
- (ii) *If $u \in L^\eta(\Omega \times \mathbb{T})$, the process $N^{K,\varepsilon}(u)$ converges to $N^K(u)$ in L^r as ε tends to 0, uniformly in $t \in \mathbb{T}$.*

Proof.

- (i) Notice first that, using the kernel's triangularity and Itô isometry,

$$\begin{aligned} \mathbb{E} |M_{t+\varepsilon}^K(u) - M_t^K(u)|^2 &= \mathbb{E} \left| \int_0^{t+\varepsilon} (K(t+\varepsilon, s) - K(t, s))u(s) dW_s + \underbrace{\int_t^{t+\varepsilon} K(t, s) dW_s}_{=0} \right|^2 \\ &= \mathbb{E} \left[\int_0^{t+\varepsilon} (K(t+\varepsilon, s) - K(t, s))^2 u(s)^2 ds \right] \end{aligned}$$

Now using Lemma 2.4, we derive the following estimates:

$$\begin{aligned} \mathbb{E} |M_t^{K,\varepsilon}(u) - M_t^K(u)|^r &= \mathbb{E} \left| \int_0^t (K(t+\varepsilon, s) - K(t, s))u(s) dW_s \right|^r \\ &\leq \mathbb{E} \left[\int_0^t (K(t+\varepsilon, s) - K(t, s))^2 u(s)^2 ds \right]^{\frac{r}{2}} \\ &\leq \mathbb{E} \left[|M_{t+\varepsilon}^K(u) - M_t^K(u)|^2 \right]^{\frac{r}{2}} \leq \mathbb{E} \left[|M_{t+\varepsilon}^K(u) - M_t^K(u)|^r \right]. \end{aligned}$$

We recall from Theorem 2.3 that for all $\gamma \in (1/r, \alpha)$,

$$\mathbb{E} \left| \sup_{s \neq t} \frac{|M_t^K(u) - M_s^K(u)|}{|t - s|^{\gamma - 1/r}} \right|^r \leq \|M^K(u)\|_{L^r(\Omega; \mathcal{C}^{\gamma - \frac{1}{r}})} \leq \|M^K(u)\|_{L^r(\Omega; \mathcal{L}_{\gamma, r})} \leq c_{\gamma, \eta} \|u\|_{L^r(\Omega \times \mathbb{T})}.$$

Hence, combining the previous inequalities we obtain

$$\mathbb{E} \left| M_t^{K, \varepsilon}(u) - M_t^K(u) \right|^r \leq \varepsilon^{\gamma r - 1} c_{\gamma, \eta} \|u\|_{L^r(\Omega \times \mathbb{T})},$$

and the first claim follows since $\gamma > 1/r$.

- (ii) We know that Ku is also γ -Hölder continuous for $u \in \mathcal{L}^\eta$. If $u \in L^\eta(\Omega \times \mathbb{T})$, then the paths $u(\omega)$ are in \mathcal{L}^η , hence $Ku(\omega)$ is also γ -Hölder continuous. Thus by similar calculations, $N_t^{K, \varepsilon}(u)$ converges to $N_t^K(u)$ in L^r as ε tends to zero. \square

3. MAIN RESULTS: STRONG SOLUTION

This section displays the proof of our main theorem so we naturally start by defining a strong solution in our context.

Definition 3.1. A solution to (1.1) is a real-valued progressively measurable stochastic process $X = \{X_t\}_{t \in \mathbb{T}}$ satisfying (1.1) almost surely and such that

$$(3.1) \quad \mathbb{P} \left(\int_0^t K(t, s)^2 \sigma(s, X_s)^2 ds < \infty, \text{ for all } t \in \mathbb{T} \right) = 1.$$

If the SVE has a unique pathwise solution, we say that it is exact.

This definition is standard [RW00, Section V.8]. If (3.1) is not satisfied then one can consider the solution up to the time of explosion. Inspired by the Yamada-Watanabe conditions [YW71a] and after space localisation, we consider the following assumptions on the coefficients:

Assumption 3.2 (Local). For each $N > 0$, there exist $C_N > 0$ and a continuous increasing function $\rho_N : (0, \infty) \rightarrow (0, \infty)$ with $\rho_N(0) := 0$ by continuity and $\int_{0+} \rho_N(u)^{-2} du = \infty$, such that for all $|x| \vee |y| < N$ and all $s \in \mathbb{T}$,

$$(3.2) \quad |\sigma(s, x) - \sigma(s, y)| \leq \rho_N(|x - y|) \quad \text{and} \quad |b(s, x) - b(s, y)| \leq C_N |x - y|.$$

Furthermore, there exists $C_G > 0$ such that for all $x \in \mathbb{R}$ and all $s \in \mathbb{T}$ the coefficients satisfy the linear growth condition

$$|b(s, x)| + |\sigma(s, x)| \leq C_G(1 + |x|).$$

As mentioned in [YW71a], the condition $\int_{0+} \rho(u)^{-2} du = \infty$ cannot be weakened. In particular, the SDE with $K_1 \equiv K_2 \equiv 1$ and $\sigma(s, x) = x^\beta$ with $\beta < \frac{1}{2}$ has an infinite number of solutions. Our last assumption concerns the kernels and is satisfied by all the kernels presented in Example 2.2.

Assumption 3.3. There exists $p \in (2, \infty)$ (hence $\frac{2p}{p-2} \in (2, \infty)$) such that,

$$\int_0^t \left(|K_1(t, s)|^2 + |K_2(t, s)|^{\frac{2p}{p-2}} \right) ds < \infty \quad \text{for all } t \in \mathbb{T}.$$

The main theorem of this paper is as follows:

Theorem 3.4. *If the two kernels K_1 and K_2 satisfy Assumptions 2.1, 2.5, 3.3 with the same parameters α and η , and the coefficients satisfy Assumption 3.2, then the stochastic Volterra equation (1.1) is exact.*

The proof of this theorem will be split into three parts. We start by proving pathwise uniqueness of the solution under stronger assumptions in Proposition 3.6, where the core of the proof resides. Then, under the same assumptions, we show the existence of a strong solution in Proposition 3.8. Finally we relax the assumptions by applying the localisation argument presented in [RW00, Theorem V.12.1]. Hence let us also consider the following global assumptions:

Assumption 3.5 (Global). There exists a continuous increasing function $\rho : (0, \infty) \rightarrow (0, \infty)$ with $\rho(0) := 0$ by continuity such that $\int_{0+} \rho(u)^{-2} du = \infty$ and a positive constant C_L such that, for all $x, y \in \mathbb{R}$ and all $s \in \mathbb{T}$:

$$(3.3) \quad |\sigma(s, x) - \sigma(s, y)| \leq \rho(|x - y|), \quad \text{and} \quad |b(s, x) - b(s, y)| \leq C_L |x - y|.$$

Finally, there exists $C_G > 0$ such that the coefficients satisfy the linear growth condition

$$(3.4) \quad |b(s, x)| + |\sigma(s, x)| \leq C_G(1 + |x|), \quad \text{for all } x \in \mathbb{R} \text{ and } s \in \mathbb{T}.$$

3.1. Uniqueness. The following proposition corresponds to Theorem 3.4, but with stronger assumptions, and is key to proving the theorem.

Proposition 3.6. *If the two kernels K_1 and K_2 satisfy Assumptions 2.1, 2.5, 3.3 with the same parameters α and η , and the coefficients satisfy Assumption 3.5, then pathwise uniqueness holds for the stochastic Volterra equation (1.1).*

The crucial point here is to check the assumptions of Theorem 2.3 on which the semimartingale approximation of Proposition 2.7 depends. Therefore the integrability condition proved in the following lemma serves two purposes: to derive the Hölder regularity of the solution and to allow the use of our convergence results.

Lemma 3.7. *Let X be a solution to (1.1), Assumptions 2.1, 3.3 for both kernels, and the linear growth condition (3.4) for the coefficients hold. Denote $r = \theta(\eta)$ then*

$$(3.5) \quad \sup_{t \in \mathbb{T}} \mathbb{E} |X_t|^{r \vee p} < \infty.$$

Moreover, X has γ -Hölder continuous paths for all $\gamma < \alpha - 1/r$.

This shows that the regularity of the solution is tied with the regularity of the kernels in the sense of Assumption 2.1.

Proof. If X solves (1.1) then the sequence of stopping times defined for $n \in \mathbb{N}$ by

$$T_n := \inf \left\{ t > 0 : \int_0^t K_2(t, s)^2 \sigma(s, X_s)^2 ds > n \right\}$$

diverges to ∞ almost surely by (3.1). Hence there exists $T_n > T$ almost surely, and

$$(3.6) \quad \mathbb{E} \left| \int_0^t K_2(t, s) \sigma(s, X_s) dW_s \right| \leq \mathbb{E} \left[\int_0^t K_2(t, s)^2 \sigma(s, X_s)^2 ds \right]^{\frac{1}{2}} \leq \sqrt{n},$$

for all $t \in \mathbb{T}$. Since $r \wedge p > 2$, then $\frac{2(r \vee p)}{r \vee p - 2} = \frac{2r}{r-2} \wedge \frac{2p}{p-2}$. By Lemma 2.4, Hölder inequality, Assumption 3.3 and the growth condition, it follows that

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^t K_2(t, s) \sigma(s, X_s) dW_s \right|^{r \vee p} \right] &\leq \mathbb{E} \left[\left| \int_0^t K_2(t, s)^2 \sigma(s, X_s)^2 ds \right|^{(r \vee p)/2} \right] \\ &\leq \left(\int_0^t |K_2(t, s)|^{\frac{2(r \vee p)}{r \vee p - 2}} ds \right)^{\frac{r \vee p - 2}{2}} \mathbb{E} \left[\int_0^t |\sigma(s, X_s)|^{r \vee p} ds \right] \\ &\leq C_{T, r} + C_r \mathbb{E} \left[\int_0^t |X_s|^{r \vee p} ds \right]. \end{aligned}$$

Similar calculations for the drift and Grönwall's lemma grant the first claim. From (3.5) and the linear growth condition, it is clear that $|b(\cdot, X)| + |\sigma(\cdot, X)| \in L^{r \vee p}(\Omega \times \mathbb{T})$ and that $\sigma(\cdot, X) \in \mathcal{A}_{K_2}$ from (3.6). Therefore Theorem 2.3 asserts that $M^{K_2}(\sigma(\cdot, X))$ has γ -Hölder continuous paths for all $\gamma < \alpha - 1/r$. Finally, since $b(\cdot, X)$ belongs in particular to \mathcal{L}^η , we get from Assumption 2.1 that $N^{K_1}b(\cdot, X)$ has the same regularity. A trivial summation yields the second claim. \square

We are now in position to prove pathwise uniqueness.

Proof of Proposition 3.6.

1) Let X and Y be two solutions on the same probability space of (1.1). For any $\varepsilon > 0$, define the semimartingales $\{X_t^\varepsilon, t \in \mathbb{T}\}$ and $\{Y_t^\varepsilon, t \in \mathbb{T}\}$ by

$$\begin{aligned} X_t^\varepsilon &:= x + \int_0^t K_1(t + \varepsilon, s) b(s, X_s) ds + \int_0^t K_2(t + \varepsilon, s) \sigma(s, X_s) dW_s, \\ Y_t^\varepsilon &:= x + \int_0^t K_1(t + \varepsilon, s) b(s, Y_s) ds + \int_0^t K_2(t + \varepsilon, s) \sigma(s, Y_s) dW_s. \end{aligned}$$

For all $\varepsilon > 0$, let $Z^\varepsilon := X^\varepsilon - Y^\varepsilon$ which is a continuous semimartingale by Lemma 2.6 and $Z := X - Y$. Hence, Proposition 2.7 implies that, for all $t \in \mathbb{T}$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} |Z_t - Z_t^\varepsilon|^{r \vee p} = 0.$$

2) Tanaka's formula [RW00, Theorem IV.43.3] for continuous semimartingales yields

$$|Z_t^\varepsilon| = \int_0^t \text{sgn}(Z_s^\varepsilon) dZ_s^\varepsilon + L_t^0(Z^\varepsilon), \quad Z_0^\varepsilon = 0,$$

where $\text{sgn}(x) = \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x \leq 0\}}$, $L_t^0(Z^\varepsilon)$ is the local time of Z^ε at 0. The identity $x^+ = \frac{1}{2}(x + |x|)$ then implies

$$(3.7) \quad (Z_t^\varepsilon)^+ = \int_0^t \mathbf{1}_{\{Z_s^\varepsilon > 0\}} dZ_s^\varepsilon + \frac{1}{2} L_t^0(Z^\varepsilon).$$

We now claim that if $\int_0^t \rho(Z_u^\varepsilon)^{-2} d[Z^\varepsilon]_u$ is finite, then $L_t^0(Z^\varepsilon) = 0$. For any $m > 0$, introduce the bounded measurable map $\varphi_m(u) := \mathbf{1}_{\{1/m, \infty\}}(u) / \rho(u)^2$. Since Z^ε is a continuous semimartingale, the Trotter-Meyer occupation density formula [RW00, Theorem 45.1] reads

$$\int_0^t \varphi_m(Z_u^\varepsilon) d[Z^\varepsilon]_u = \int_{\mathbb{R}} \varphi_m(a) L_t^a(Z^\varepsilon) da,$$

and letting m tend to infinity, the monotone convergence theorem implies that

$$\int_0^t \frac{d[Z^\varepsilon]_u}{\rho(u)^2} = \int_0^\infty \frac{L_t^a(Z^\varepsilon)}{\rho(u)^2} da.$$

The local time function $a \mapsto L_t^a(Z^\varepsilon)$ is right continuous, and therefore the right-hand side diverges unless $L_t^0(Z^\varepsilon) = 0$ because of the behaviour of the function ρ around the origin in Assumption 3.5. We now prove that

$$(3.8) \quad \begin{aligned} \int_0^t \frac{d[Z^\varepsilon]_u}{\rho(Z_u^\varepsilon)^2} &= \int_0^t \frac{(\sigma(u, X_u) - \sigma(u, Y_u))^2}{\rho(Z_u^\varepsilon)^2} K(u + \varepsilon, u)^2 du \\ &\leq \int_0^t \frac{\rho(Z_u)^2}{\rho(Z_u^\varepsilon)^2} K_2(u + \varepsilon, u)^2 du. \end{aligned}$$

is indeed finite, at least for all ω in the set Ω^ε which we define now.

3) For any $t \in \mathbb{T}$, introduce the measurable spaces

$$\mathcal{N}_t^\varepsilon := \left\{ \omega \in \Omega : Z_t^\varepsilon(\omega) = 0 \text{ and } Z_t(\omega) \neq 0 \right\},$$

as well as $\mathcal{N}^\varepsilon := \bigcup_{t \in \mathbb{T}} \mathcal{N}_t^\varepsilon$ and $\Omega^\varepsilon := \Omega \setminus \mathcal{N}^\varepsilon$. Since Z_t^ε converges to Z_t in L^2 and both processes are continuous, there exists a subsequence (which we denote again by ε) such that Z_t^ε converges to Z_t for all $t \in \mathbb{T}$ almost surely. Hence, notice that by Fatou's lemma and because $\limsup_n \mathbf{1}_{A_n} = \mathbf{1}_{\limsup_n A_n}$:

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\mathcal{N}^\varepsilon) &= \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(\left\{ \omega \in \Omega : \exists t \in \mathbb{T}, Z_t^\varepsilon(\omega) = 0 \text{ and } Z_t(\omega) \neq 0 \right\}\right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(\left\{ \omega \in \Omega : \exists t \in \mathbb{T}, Z_t^\varepsilon(\omega) \neq Z_t(\omega) \right\}\right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(\left\{ \omega \in \Omega : \|Z^\varepsilon(\omega) - Z(\omega)\|_\infty > 0 \right\}\right) \\ &\leq \mathbb{P}\left(\limsup_{\varepsilon \rightarrow 0} \left\{ \omega \in \Omega : \|Z^\varepsilon(\omega) - Z(\omega)\|_\infty > 0 \right\}\right) \\ &= \mathbb{P}\left(\forall \varepsilon > 0, \exists \delta \in (0, \varepsilon) : \|Z^\delta - Z\|_\infty > 0\right) \\ &\leq \mathbb{P}\left(\limsup_{\varepsilon \rightarrow 0} \|Z^\varepsilon - Z\|_\infty > 0\right) = 0. \end{aligned}$$

Going back to the estimate (3.8), on Ω^ε , either $\rho(Z_u)^2 = 0$ or $\rho(Z_u^\varepsilon)^{-2}$ is finite for all $u \in \mathbb{T}$, and therefore no blow-up can occur, so that $\int_0^t \rho(Z_u^\varepsilon(\omega))^{-2} d[Z^\varepsilon(\omega)]_u$ is finite and $L_t^0(Z^\varepsilon(\omega)) = 0$ for all $\varepsilon > 0$ and all $\omega \in \Omega^\varepsilon$.

4) Finally, from (3.7), Cauchy-Schwarz and Assumption 3.5 imply

$$\begin{aligned} \mathbb{E}[(Z_t^\varepsilon)^+] &= \mathbb{E}[(Z_t^\varepsilon)^+ \mathbf{1}_{\mathcal{N}^\varepsilon}] + \mathbb{E}[(Z_t^\varepsilon)^+ \mathbf{1}_{\Omega^\varepsilon}] \\ &= \mathbb{E}[(Z_t^\varepsilon)^+ \mathbf{1}_{\mathcal{N}^\varepsilon}] + \mathbb{E}\left[\int_0^t \mathbf{1}_{\{Z_s^\varepsilon > 0\}} K_1(t + \varepsilon, s) (b(s, X_s) - b(s, Y_s)) ds \mathbf{1}_{\Omega^\varepsilon}\right] \\ &\leq \mathbb{E}[(Z_t^\varepsilon)^+ \mathbf{1}_{\mathcal{N}^\varepsilon}] + \left[\int_0^t K_1(t + \varepsilon, s)^2 ds\right]^{\frac{1}{2}} \left[\int_0^t \mathbb{E}[C_L Z_s \mathbf{1}_{\{Z_s^\varepsilon > 0\}} \cap \Omega^\varepsilon]^2 ds\right]^{\frac{1}{2}}, \end{aligned}$$

where the first term tends to zero because Z_t^ε is integrable. Moreover,

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \mathbb{E}[Z_s \mathbf{1}_{\{Z_s^\varepsilon > 0\}} \cap \Omega^\varepsilon]^2 ds = \int_0^t \mathbb{E}[Z_s \mathbf{1}_{\{Z_s > 0\}}]^2 ds = \int_0^t \mathbb{E}[Z_s^+]^2 ds,$$

by (twice) dominated convergence, and we finally conclude that

$$(3.9) \quad \mathbb{E} [Z_t^+]^2 = \lim_{\varepsilon \rightarrow 0} \mathbb{E} [(Z_t^\varepsilon)^+]^2 \leq C_{T,L} \int_0^t \mathbb{E} [Z_s^+]^2 ds,$$

which means $\mathbb{E} [Z_t^+] = 0$ by Grönwall's inequality, i.e. $X_t \geq Y_t$ almost surely. Interchanging the roles of X and Y reverses the inequality and the claim follows. \square

3.2. Existence. As we mentioned in the introduction, the kernels present in our SVE are too general for known weak existence results. Hence we undertake to prove the existence of a strong solution using the traditional Picard iteration and calculations similar to the uniqueness proof.

Proposition 3.8. *Under the same assumptions as Proposition 3.6, the SVE (1.1) has a strong solution.*

Some preliminaries are needed before getting to the proof of this result. We consider the Banach space \mathbb{L}_T^r of all progressively measurable processes X such that $\|X\|_{L^r(\Omega \times \mathbb{T})}$ is finite, and the map $X \mapsto I(X)$ from \mathbb{L}_T^r to itself, defined as

$$(3.10) \quad I(X)_t := x + \int_0^t K_1(t,s)b(s, X_s) ds + \int_0^t K_2(t,s)\sigma(s, X_s)dW_s, \quad \text{for } t \in \mathbb{T}.$$

Lemma 3.9. *Under the same assumptions as Proposition 3.6, the map I is well-defined from \mathbb{L}_T^r to itself and $\mathbb{E} [\sup_{t \in \mathbb{T}} |I(X)_t|^r] < \infty$, for any $X \in \mathbb{L}_T^r$.*

Proof. By Theorem 2.3, there exists $\gamma > 0$ such that $I(X)$ has γ -Hölder continuous paths. Thus there exists $A \in L^r$ such that for all $t, t' \in \mathbb{T}$,

$$|I(X)(\omega)_t - I(X)(\omega)_{t'}| \leq A(\omega) |t - t'|, \quad \text{for all } \omega \in \Omega.$$

In particular $\sup_{t \in \mathbb{T}} |I(X)(\omega)_t|^r \leq A(\omega) T^{r\gamma}$. This yields the existence of $C_{T,r} > 0$ such that $\mathbb{E} [\sup_{t \in \mathbb{T}} |I(X)_t|^r] \leq C_{T,r}$, and therefore $I(X) \in \mathbb{L}_T^r$ for all $X \in \mathbb{L}_T^r$. \square

Now we are all set to prove the main result of this subsection.

Proof of Proposition 3.8.

1) Thanks to Lemma 3.9 we can define by iteration the sequence, in \mathbb{L}_T^r , $X^0 = x$ and $X^{(n+1)} := I(X^{(n)})$ for all $n \geq 0$, such that $X^{(n)} \in \mathbb{L}_T^r$ for each $n \geq 0$, and

$$(3.11) \quad \sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{t \in \mathbb{T}} |X_t^{(n)}|^r \right] < \infty.$$

We now prove that $\{X^{(n)}\}$ is a Cauchy sequence in a space invariant for the mapping I , and for which Tanaka's formula holds. To this end, introduce the Banach space $\mathbb{L}_T \subset \mathbb{L}_T^1$ endowed with the norm $\|X\|_{\mathbb{L}_T} := \int_0^T \mathbb{E}[|X_t|]^2 dt$, and we prove convergence in \mathbb{L}_T even though the sequence $\{X^{(n)}\}_{n \in \mathbb{N}}$ belongs to \mathbb{L}_T^r .

2) For each n , define the sequence of semimartingales $\{X^{(n,\varepsilon)}\}_{\varepsilon > 0}$ in \mathbb{L}_T^r by

$$X_t^{(n,\varepsilon)} := x + \int_0^t K_1(t+\varepsilon, s)b(s, X_s^{(n)}) ds + \int_0^t K_2(t+\varepsilon, s)\sigma(s, X_s^{(n)}) dW_s,$$

for $t \in \mathbb{T}$. From Proposition 2.7, $\lim_{\varepsilon \rightarrow 0} \sup_{t \in \mathbb{T}} \mathbb{E} \left| X_t^{(n,\varepsilon)} - X_t^{(n)} \right|^r = 0$, and we can use Tanaka's formula and similar calculations to the uniqueness proof for the local time. For $Z^{(n,m)} := X^{(n)} - X^{(m)}$, the analogue to (3.9) yields

$$\begin{aligned} \mathbb{E} \left[\left| Z_t^{(n+1,m+1)} \right|^2 \right] &= \left\{ \mathbb{E} \left[(Z_t^{(n+1,m+1)})^+ \right] + \mathbb{E} \left[(-Z_t^{(n+1,m+1)})^+ \right] \right\}^2 \\ &\leq C_{T,L} \int_0^t \mathbb{E} \left[\left| Z_s^{(n,m)} \right|^2 \right] ds. \end{aligned}$$

The sequence $\Delta_t^{(n,m)} := \|Z^{(n,m)}\|_{\mathbb{L}_t}$ satisfies

$$\Delta_t^{(n+1,m+1)} \leq \int_0^t C_{T,L} \int_0^s \mathbb{E} \left[\left| Z_u^{(n,m)} \right|^2 \right] du ds = C_{T,L} \int_0^t \Delta_s^{(n,m)} ds.$$

Let $\Delta_t := \limsup_{n,m \rightarrow \infty} \Delta_t^{(n,m)}$ and observe from (3.11) that $\sup_{n,m \in \mathbb{N}} \sup_{t \in \mathbb{T}} \Delta_t^{(n,m)}$ is finite. Hence we can use Fatou's lemma to deduce $\Delta_t \leq C_{T,L} \int_0^t \Delta_s ds$, and Grönwall's inequality yields $\Delta_t = 0$ for all $t \in \mathbb{T}$. Therefore, $\{X^{(n)}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{L}_T and there exists a limit $\bar{X} \in \mathbb{L}_T$ such that

$$\lim_{n \rightarrow \infty} \int_0^T \mathbb{E} \left[\left| X_t^{(n)} - \bar{X}_t \right|^2 \right] dt = 0.$$

3) To show that $\bar{X} = I(\bar{X})$ almost surely we recall from previous calculations that

$$\begin{aligned} \int_0^T \mathbb{E} \left[\left| \bar{X}_t - I(\bar{X})_t \right|^2 \right] dt &= \lim_{n \rightarrow \infty} \int_0^T \mathbb{E} \left[\left| X_t^{(n+1)} - I(\bar{X})_t \right|^2 \right] dt \\ &\leq \lim_{n \rightarrow \infty} \left\{ \int_0^T C_{T,L} \int_0^t \mathbb{E} \left[\left| X_s^{(n)} - \bar{X}_s \right|^2 \right] ds dt \right\} = 0, \end{aligned}$$

and therefore \bar{X} is a fixed point of I and a solution of (1.1). \square

Note that even though the convergence took place in \mathbb{L}_T , we know from Lemmas 3.7 or 3.9 that $\bar{X} \in \mathbb{L}_T^r$.

3.3. Localisation. The first two parts of the proof of Theorem 3.4 have now been established. Therefore we can use the localisation argument to relax our assumptions in the sense of 3.2 and finish the main proof.

Proof of Theorem 3.4. This proof is inspired by that of [RW00, Chapter V, Theorem 12.1]. We introduce the families of coefficients $\{\sigma_N\}_{N \geq 0}$ and $\{b_N\}_{N \geq 0}$ such that, for all $N \geq 0$, they satisfy Assumption 3.5 and agree with σ and b respectively on $\{(t, x) : t \vee x \leq N\}$. This way, for each N , Propositions 3.6 and 3.8 guarantee that there exists a unique progressively measurable process $\{X_t^N\}_{t \in \mathbb{T}}$ satisfying (1.1). Define now the increasing stopping time, for each $N \geq 0$,

$$\tau_N := \inf\{t \geq 0 : |X_t^N| \geq N\} \wedge N,$$

and the process $X = X^N$ on $[0, \tau_N]$ It yields, for all $t \in [0, \tau_N]$,

$$\begin{aligned} X_t &= X_t^N = x + \int_0^t K_1(t, s) b_N(X_s^N) ds + \int_0^t K_2(t, s) \sigma_N(X_s^N) dW_s \\ &= x + \int_0^t K_1(t, s) b(s, X_s) ds + \int_0^t K_1(t, s) \sigma(s, X_s) dW_s. \end{aligned}$$

Recall from Lemma 3.7 that there exists $\gamma > 0$ such that X has γ -Hölder continuous paths. Therefore there exists $A \in L^r$ such that for all $t, t' \in \mathbb{T}$,

$$|X_t(\omega) - X_{t'}(\omega)| \leq A(\omega) |t - t'|, \quad \text{for all } \omega \in \Omega.$$

In particular $\sup_{t \in \mathbb{T}} |X_t(\omega)|^r \leq A(\omega) T^{r\gamma}$. This yields the existence of a constant $C_{T,r}$ such that

$$\mathbb{E} \left[\sup_{t \in \mathbb{T}} |X_t|^r \right] \leq C_{T,r},$$

and hence for all $N \geq 0$,

$$\mathbb{P}(\tau_N < T) = \mathbb{P} \left(\sup_{t \in \mathbb{T}} |X_t| > N \right) \leq N^{-r} \mathbb{E} \left[\sup_{t \in \mathbb{T}} |X_t|^r \right]$$

converges to zero as N tends to infinity. Therefore $\mathbb{P}(\sup \tau_N = \infty) = 1$, and X is a solution on \mathbb{T} . Finally to prove uniqueness consider X and X' both satisfying (1.1) with coefficients obeying Assumption 3.2. As we just showed, they must be equal to X^N almost surely on $[0, \tau_N]$, hence $X = X'$ almost surely on \mathbb{T} . \square

4. APPLICATION TO ROUGH HESTON AND MULTI-DIMENSIONAL EXTENSION

4.1. The rough Heston model. Recently, El Euch and Rosenbaum [EER19] proposed a rough version of the classical Heston model, widely used in the financial industry, in order to capture the specificities of Equity options markets, and weak existence and uniqueness was derived in [ALP17]. A generalised version with time-dependent drift was also introduced in [EER18] and takes the form

$$(4.1) \quad \begin{aligned} Y_t &= y_0 - \frac{1}{2} \int_0^t X_s ds + \int_0^t \sqrt{X_s} d(\bar{\rho} B_s + \rho W_s), \\ X_t &= x_0 + \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \lambda(\theta(s) - X_s) ds + \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \xi \sqrt{X_s} dW_s, \end{aligned}$$

where $y_0 \in \mathbb{R}$, $x_0, \xi, \lambda > 0$, $\rho \in (-1, 1)$, $\bar{\rho} := \sqrt{1 - \rho^2}$, $H \in (0, 1)$, W and B are independent Brownian motions, and $\theta : \mathbb{T} \rightarrow \mathbb{R}_+$ is some deterministic function. Here H corresponds to the Hurst exponent of the fractional Brownian motion and governs the Hölder regularity of the solution. The combination of the square root diffusion coefficient and the singular kernel seriously complicates the study of certain aspects of this model, as we have seen. The process Y represents the log-price of an asset while X corresponds to its squared instantaneous volatility. Since the log stock price process is represented as an integrated version of the volatility, it only suffices to prove existence and uniqueness of the latter.

Proposition 4.1. *The system (4.1) is exact.*

Proof. The diffusion coefficient for X in (4.1) is only defined on \mathbb{R}_+ so that we cannot apply Theorem 3.4 directly as it would require to build a strong non-negative solution. The kernel satisfies Assumption 2.1 by [Dec02, Theorem 4.1] and it is easy to check that it also satisfies Assumptions 2.5 and 3.3. Clearly, $b(s, x) = \lambda(\theta(s) - x)$ is Lipschitz continuous and $\tilde{\sigma}(s, x) := \xi \sqrt{x^+}$ is $\frac{1}{2}$ -Hölder continuous therefore they satisfy Assumption 3.5. This implies that pathwise uniqueness holds for the SVE for X in (4.1) with coefficients b and $\tilde{\sigma}$ by Proposition 3.6. Any solution of the original SVE must be non-negative, in which case $\sigma = \tilde{\sigma}$, hence pathwise uniqueness holds for this equation too. Moreover, a non-negative weak solution of the second

SDE in (4.1) was constructed in [ALP17, Theorem 3.6], and therefore this SVE is exact. Plugging this solution into the first component of (4.1) yields the claim. \square

Remark 4.2. Looking at Assumption 2.1 we have $H = \alpha$ and one can choose any $\eta < 2$ therefore $r = \theta(\eta)$ can be as large as one wants. This means, by Proposition 3.7, that V is almost surely γ -Hölder continuous for any $\gamma < H$. Hence we also retrieve the Hölder continuity proved in [EER18]. Furthermore this reasoning applies to any strong solution of an SVE with the same kernel, regardless of the form of the coefficients.

A direct consequence is the pathwise uniqueness of the forward variance curve $\{\mathbb{E}[V_t|\mathcal{F}_s]\}_{t \geq s}$ for any $s \in \mathbb{T}$ and of the option price process $\{C_t := \mathbb{E}[g(X_\cdot)|\mathcal{F}_t]\}_{t \in \mathbb{T}}$ for some measurable function $g : \Omega \rightarrow \mathbb{R}$, which consolidates the theoretical setup of the hedging strategy derived in [EER18].

4.2. Multi-dimensional version. Tanaka's formula and Yamada-Watanabe's theorem only hold in one dimension. A proper multidimensional pathwise uniqueness version for SDEs with non-Lipschitz coefficients represents a complex challenge [YW71b], and only few limited extensions exist [Swa02]. Counterexamples to the weak uniqueness were displayed in [YW71b, Swa01], while the Yamada-Watanabe approach fails in several dimensions because of the mutual dependence between the components. However our one-dimensional theorem (Theorem 3.4) can be extended to the multidimensional case as long as the coefficient's components do not depend *mutually* on each other. Consider

$$(4.2) \quad \mathbf{X}_t = \mathbf{x} + \int_0^t \mathbf{K}_1(t, s) \cdot \mathbf{b}(s, \mathbf{X}_s) ds + \int_0^t \mathbf{K}_2(t, s) \cdot \boldsymbol{\sigma}(s, \mathbf{X}_s) d\mathbf{W}_s, \quad t \in \mathbb{T},$$

where \cdot represents component-wise multiplication, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{K}_1, \mathbf{K}_2 : \mathbb{T}^2 \rightarrow \mathbb{R}^d$ are multidimensional kernels, $\mathbf{b} : \mathbb{T} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\boldsymbol{\sigma} : \mathbb{T} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$ are Borel-measurable functions, and \mathbf{W} is an m -dimensional Brownian motion. Both functions are assumed progressive in the following sense:

$$\mathbf{b}(s, \mathbf{x}) = \begin{bmatrix} b_1(s, x_1) \\ b_2(s, x_1, x_2) \\ \vdots \\ b_d(s, x_1, \dots, x_d) \end{bmatrix}, \quad \boldsymbol{\sigma}(s, \mathbf{x}) = \begin{bmatrix} \sigma_{11}(s, x_1) & \cdots & \sigma_{1m}(s, x_1) \\ \sigma_{21}(s, x_1, x_2) & \cdots & \sigma_{2m}(s, x_1, x_2) \\ \vdots & \ddots & \vdots \\ \sigma_{d1}(s, x_1, \dots, x_d) & \cdots & \sigma_{dm}(s, x_1, \dots, x_d) \end{bmatrix},$$

such that the dependence only goes in one direction. Thus we can prove uniqueness for the first line and then to the others by induction.

Corollary 4.3. *Assume that 2.1, 2.5 3.3 hold for all kernels and that each element b_i, σ_{ij} , $1 \leq i \leq d$, $1 \leq j \leq m$, satisfy Assumption 3.2 where the Lipschitz constant and the diffusion modulus can be different from row to row (may vary for different i but are the same for different j). Then the SVE (4.2) is exact.*

Proof. The first element X^1 of \mathbf{X} is one-dimensional, so that the proof in the one-dimensional case is not altered by the additional diffusion terms:

$$X_t^1 = x^1 + \int_0^t K_1^1(t, s) b_1(s, X_s^1) ds + \sum_{j=1}^m \int_0^t K_2^1(t, s) \sigma_{1j}(s, X_s^1) dW_s^j.$$

Therefore strong existence and uniqueness hold for X^1 . If strong existence and uniqueness stands for all $X^i, i < k$ then the same holds for X^k by plugging the previous elements in the coefficients of the k -th row. More precisely,

- the integrability conditions are still ensured by the linear growth condition.
- For the existence we mimic the proof of Proposition 3.8: define the Cauchy sequence $\mathbf{X}^{(0)} = (X^1, \dots, X^{k-1}, x)^\top$ and $\mathbf{X}^{(n+1)} = (X^1, \dots, X^{k-1}, I(X^{(n)})^k)^\top$ for all $n \geq 1$ on \mathbb{L}_T . Hence the difference between $\mathbf{X}^{(n)}$ and $\mathbf{X}^{(m)}$ does not depend on previous elements (since they are fixed). Therefore the existence proof becomes one-dimensional and this has already been dealt with.
- The uniqueness follows the same pattern where, for any solutions \mathbf{X} and \mathbf{Z} ,

$$\begin{aligned} |b_k(X_t^1, \dots, X_t^k) - b_k(Z_t^1, \dots, Z_t^k)| &\leq C_L |\mathbf{X}_t - \mathbf{Z}_t| = C_L |X_t^k - Z_t^k|, \\ |\sigma_{kj}(X_t^1, \dots, X_t^k) - \sigma_{kj}(Z_t^1, \dots, Z_t^k)| &\leq \rho(|\mathbf{X}_t - \mathbf{Z}_t|) = \rho(|X_t^k - Z_t^k|), \end{aligned}$$

almost surely, which reduces to the one-dimensional proof of Theorem 3.6.

- Finally, the localisation procedure does not suffer from the multidimensionality and can be applied directly.

□

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